Vectors

Introduction

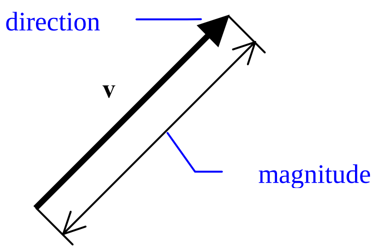
This paper covers a general description of vectors first (as can be found in mathematics books) and will stray into the more practical areas of graphics and animation.
Anyone working in graphics subjects such as animation or CAD design will find the information within very useful.

Definition

Vectors indicate a quantity (like velocity or force) that has both magnitude and direction. Vectors have both a graphical and algebraic representation. Graphically, the vector is drawn as an arrow pointing into a particular direction, its length representing the magnitude of the quantity (such as mass, speed, length, etc). Algebraically, vectors are written as a set of coordinates.
The fact that vectors have direction makes them very useful in 3D graphics. We do operations on vectors and matrices to rotate, scale, translate objects and do things such as finding intersections, perpendiculars to planes, …
The great thing about vectors is that you can both visualize and calculate the physical entities. I will use CAD to illustrate many of the concepts discussed within.

The Two Components of a Vector

Names of the vectors (labels) will be written in bold (**v** in the example). This is conforming to current norms, though many other representations exist. If written notes are used, it is customary to draw a little arrow or line above or below the label.

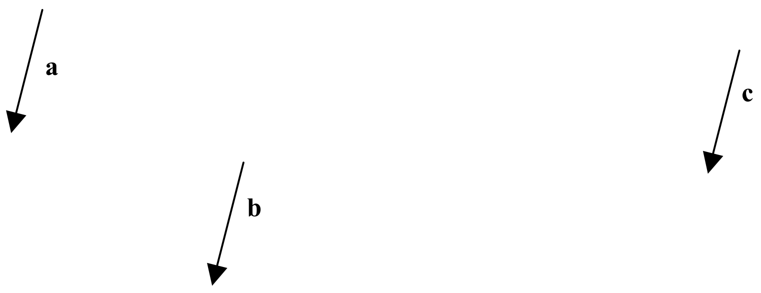


$$\mathbf{v} = \overline{v} = \underline{v} = \vec{v}$$

The magnitude (length) of a vector is represented by a single value, referred to as a **scalar**. This is true regardless of the number of dimensions we are working in (see later).

Direction is more complicated; you need at least two values in order to define a direction. The number of values needed to define direction will always correspond to the number of dimensions we are working in. Obviously, once we are working in more than 3 dimensions, graphical representation will become very difficult. Nevertheless the mathematics used will follow the same logic. We will see that Complex Numbers are a convenient way to represent both direction and magnitude of vectors.

Note that the location of a vector is not relevant. For example, vectors **a**, **b** and **c** in the figure below are identical as long as their magnitude and direction stay the same. Therefore, **a = b = c**.



We can represent the magnitude of a vector using the *absolute* symbol since it is the length of the vector, i.e. the magnitude of vector **a** above is represented by

|**a**|

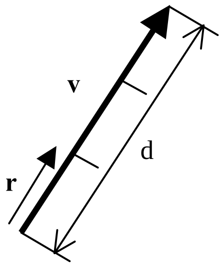
Since the three vectors are equal, we can say that

|**a**| = |**b**| = |**c**|

Unit vectors and multiplying vectors with a scalar value

Unit vectors are vectors that have a magnitude of 1 (unity).
Any vector, **v**, can be expressed as the product of its magnitude and a unit vector in the same direction.

Assume unit vector **r** (of length 1) and vector **v**



d = |**v**|

v = |**v**| * **r**

In the figure, **v** = 3**r**

Unit Direction vectors

In the last figure, the vector **r** also indicates direction. So to be more exact, we should call **r** the **Unit Direction Vector** for **v**.

When naming unit direction vectors for different vectors, they often bare the symbol of the vector they refer to and 'cap' this symbol with a '^' to distinguish them from the vectors they refer to:

$r = \hat{v}$

Component Equation of the vector

So now we can represent the two components of a vector symbolically. The unit direction vector establishes the length of one unit (whatever that is) and gives it direction, it is the vector's measuring stick. The magnitude is represented by the length (absolute value) of the vector.

The component equation clearly shows this interaction:

$\mathbf{v} = \hat{\mathbf{v}} * |\mathbf{v}|$

$\hat{\mathbf{v}}$ is the vector's **directional** component with a unit of 1 (the unit direction vector)
|v| is the vector's **scalar** component (its magnitude or length)
v is the vector in its totality (interaction of both direction and magnitude)

Normalisation and Normals

Last equation also shows how we can make any vector into a unit vector by dividing the vector by its length:

$$\hat{\mathbf{v}} = \frac{v}{|v|}$$

The process of doing this is called **normalisation**.

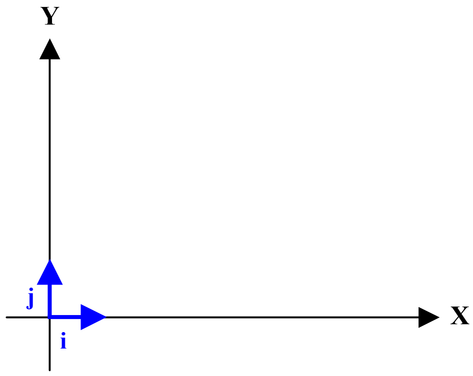
Normalisation is used in those situations where the vector's length (magnitude) will otherwise interfere with the calculations.

There is some confusion about the term *normals*. A *normal* is a vector that is perpendicular to 2 other vectors (we will later see that it is the result of the *cross product* operation). However, this normal may have any length. So we can *normalise* the *normal* to a length of 1, just as we can normalise any vector to a unit of 1.

Complex numbers

Complex numbers are the ideal medium to represent direction.
Consider 2 unit direction vectors in the case of 2D:

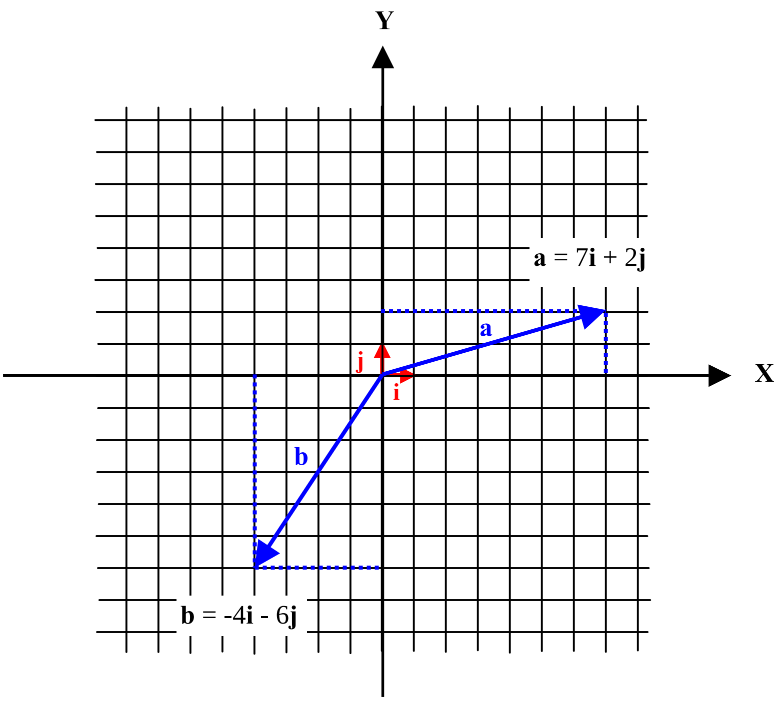
i, pointing in the direction of the X axis
j, pointing in the Y direction



Though we only work in 2 dimensions for illustrative purposes, the argument easily extends to any number of dimensions, each successive one perpendicular to the former.

As stated before, once more than 3 dimensions are used, graphical representation becomes difficult and the argument can only be extended theoretically, though in the same simple and logical way.

We can now represent any direction by using the direction unit vectors **i** and **j** as measuring sticks.



Distance

Note that we can apply Pythagoras' right angle triangle equation to determine the length of the vectors

$$|a| = \sqrt{7^2 + 2^2} = \sqrt{53}$$

$$|b| = \sqrt{(-4)^2 + (-6)^2} = \sqrt{52}$$

Matrix notation and the Cartesian Coordinate system

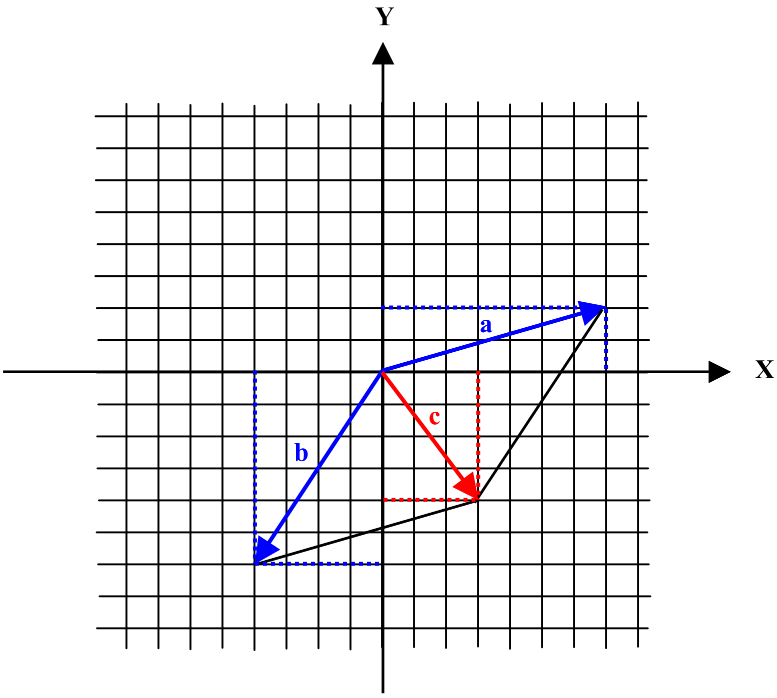
A vector can be represented by using the *Cartesian Coordinate System*. This system is used in graphics and animation applications. It is somewhat like using the complex number system without using the unit direction vectors **i** and **j** (they are implied).
We will use the matrix notation for the vectors. The vector, after all, is a one dimensional matrix. I prefer to use the vertical vector notation because it clearly isolates the coordinates, making them easier to read.

$$\mathbf{a} = \begin{bmatrix} 7 & 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

It is important to realise that all vectors thus notated have their origin at point (0,0).

Adding vectors

We can add vectors by adding their coordinates.

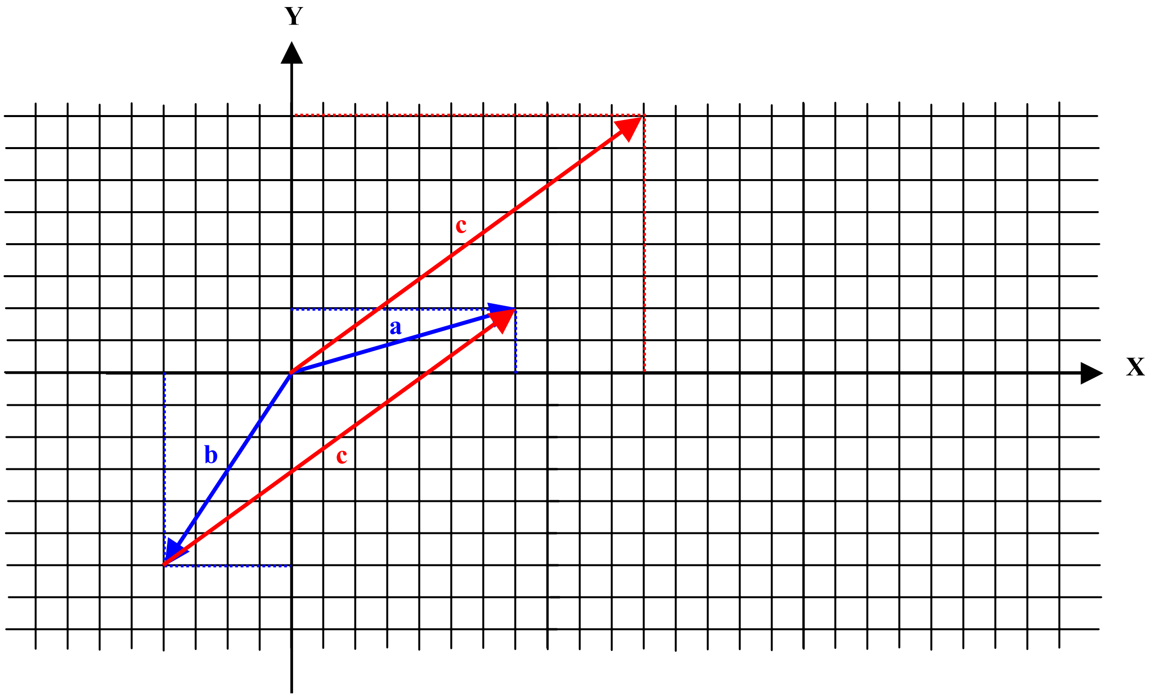


$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

$$\mathbf{c} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Note that graphically, we just construct a parallelogram.

Subtracting vectors



Subtracting vectors gives you the difference between the 2 vectors. Note that both **c** vectors are the same; one is derived graphically while the other has been calculated resulting in the coordinates below. This calculated vector starts at the origin (0,0), but shows the same direction and magnitude as the graphical-derived one. Therefore both these vectors are equal.

$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$

$$\mathbf{c} = \begin{bmatrix} \mathbf{7} \\ \mathbf{2} \end{bmatrix} - \begin{bmatrix} \mathbf{-4} \\ \mathbf{-6} \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

Position vectors

Any vector that has its origin in (0,0) is a position vector. The blue vectors and one of the red vectors in the figure above are position vectors.

The following excerpt is from a study book written by Chris Harman and Patricia Cretchley, associate professors at the Faculty of Sciences in The University of Southern Queensland.

The Equation of a Plane in R3:

Consider a particular plane in space: imagine a flat sheet extending without bound in any direction. We seek a property that distinguishes all the points on that particular plane from other points in space.
A line's direction is characterised by a direction vector parallel to it, because a line has only one such direction. A plane's direction cannot be characterized in that way by a vector parallel to it, however, because many vectors with different directions are parallel to it: consider any vector that joins two points in the plane, for example.

But there is only one direction perpendicular to the plane. *Imagine a table-top or a slanted roof: if we drew vectors perpendicular to the surface they would all be parallel, though some might point in opposite directions. We say all such vectors are normal to the plane, meaning perpendicular, and we call any one of them a* ***normal*** *for that plane.*

A plane has many normals: if you have one, any positive or negative multiple will also be normal to that plane.

- *Parallel planes have parallel normals.*
- *Planes that are not parallel will have normals that are not parallel.*
- *Planes that are perpendicular, ie orthogonal, have normals that are orthogonal.*
- *Given a normal, there are infinitely many planes, all parallel, that are perpendicular to it. To distinguish one of them in particular, we need to know at least one point on it. So a plane is fully specified by knowing*
    - *a vector perpendicular or normal to it,*
    - *and a particular point on it.*
- *This information should therefore be enough for us to arrive at an equation*
- *that characterises all its points.*

Suppose, therefore, that point $P(x_0, y_0, z_0)$ lies on a plane and that vector _n = (a, b, c) is normal or perpendicular to plane.

Let us show **n** *perpendicular to the plane at the given point P*
Now suppose Q(x, y, z) is a variable point anywhere on that plane.
Join Q to P. This gives vector PQ lying in the plane. Since **n** *is normal to the plane,* **n** *will be perpendicular to PQ. This is only true if Q is on the plane: if Q is above or below the plane, joining it to P will make an acute or obtuse angle with* **n**.

So for all points Q on the plane, and no others, PQ is perpendicular to **n**.
Using the dot product test for orthogonality, therefore, we have an equation that characterises points Q on the plane:

$PQ \cdot \mathbf{n} = 0.$

Giving $P(x_0, y_0, z_0)$ and Q(x, y, z) their coordinates, we get

$[(x, y, z) - (x_0, y_0, 0)] \cdot \mathbf{n} = 0.$

Using the distributive rule on the bracket, gives yet another form:

$(x, y, z) \cdot \mathbf{n} - (x_0, y_0, z_0) \cdot \mathbf{n} = 0,$

and putting the second term on the RHS gives

$(x, y, z) \cdot \mathbf{n} = (x_0, y_0, z_0) \cdot \mathbf{n}$

Vectors in the Plane and in Space

These are all different forms of the equation for the plane, and give us different ways of finding it. Notice in particular the easy method:

(variable point) · (normal) = (fixed point) · (normal)

Vector **n** *= (a, b, c), so this gives*

$(x, y, z) \cdot (a, b, c) = (x_0, y_0, z_0) \cdot (a, b, c)$,

and taking dot products gives

$ax + by + cz = ax_0 + by_0 + cz_0$.

The LHS cannot be simplified because x, y and z are variables. But the RHS will reduce to an answer, d, say.
So in general the equation of a plane has form

$ax + by + cz = d$.

Note *also that in this form, the coefficients (a, b, c) give a normal to the plane.*

Example:

1. The plane that passes through the point (1, 2,−3) in such a way that it is normal to the vector (2,−1, 4), has equation $(x, y, z) \cdot (2,-1, 4) = (1, 2,-3) \cdot (2,-1, 4.)$
Evaluating the dot products gives $2x - y + 4z = 2- 2 - 12$, *or simply* $2x - y + 4z = -12$.

To find points (x, y, z) on this plane, we can substitute values for any two of the variables, and work out the third. For example, substituting easy values x = 0 and y = 0, we find 4z = −12, so that z = −3. The point (0, 0,−3) therefore lies on that plane.

2. Sometimes the normal is not given, but can be gleaned from other information: eg, if the plane is known to be parallel to another plane with given equation, then one can use the known one's normal (ie its coefficients of x, y and z) as a normal for the required plane;

3. Note that ***cross product can be used to create a normal to a plane****, from two vectors that lie in the plane or are parallel to it. For example, given three points P,Q and R in the plane (not all on the same straight line) joining two of them in any direction gives a vector that lies in the plane. Joining any other pair will make another vector in the plane. The cross product of these, say* $PQ \times RQ$, *gives a vector perpendicular to those two vectors, and hence also to the plane in which they lie.*

Finding determinants for matrices of any size

This article assumes basic knowledge about matrices.

There is a simple procedure to finding the determinant …

Example

$$M = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 5 & -6 & 0 & 2 \\ 8 & 7 & 3 & 1 \\ 0 & 0 & 5 & -1 \end{bmatrix}$$

Choose any row or column. In this case the 4th row contains 2 zeros, choosing this row will make calculation easier…

Consider also the 4x4 'alternating signature' table …

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

signatures applied to last row in matrix M

These signatures will be applied to the corresponding elements of the matrix. In our case, looking at the last row, the value 5 becomes -5 and value -1 remains -1. Note that only the negative values have an impact on the calculations.

The minus signatures thus allocated will be circled red in the following figures and the actual values in the row of the matrix will be circled green.

Draw lines through each non-zero element of this last row and form *minors* (sub-matrices) by writing down the elements NOT crossed by the lines …

$$M = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 5 & -6 & 0 & 2 \\ 8 & 7 & 3 & 1 \\ 0 & 0 & 5 & -1 \end{bmatrix} \qquad -5 * \begin{bmatrix} 3 & 2 & 4 \\ 5 & -6 & 2 \\ 8 & 7 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 5 & -6 & 0 & 2 \\ 8 & 7 & 3 & 1 \\ 0 & 0 & 5 & -1 \end{bmatrix} \qquad -1 * \begin{bmatrix} 3 & 2 & 1 \\ 5 & -6 & 0 \\ 8 & 7 & 3 \end{bmatrix}$$

So far we have

$$-5*\begin{bmatrix}3 & 2 & 4\\5 & -6 & 2\\8 & 7 & 1\end{bmatrix}-1*\begin{bmatrix}3 & 2 & 1\\5 & -6 & 0\\8 & 7 & 3\end{bmatrix}$$

Repeat the process in a recursive fashion

The signature table for a 3x3 matrix is

$$\begin{bmatrix}+ & - & +\\- & + & -\\+ & - & +\end{bmatrix}$$

signatures applied to first row of first minor

signatures applied to second row of second minor

I choose 1st row in the first matrix and 2nd row in the second matrix…

First row

$$-5*\begin{bmatrix}3 & 2 & 4\\5 & -6 & 2\\8 & 7 & 1\end{bmatrix} \qquad -5*\begin{bmatrix}3 & 2 & 4\\5 & -6 & 2\\8 & 7 & 1\end{bmatrix} \qquad -5*\begin{bmatrix}3 & 2 & 4\\5 & -6 & 2\\8 & 7 & 1\end{bmatrix}$$

$$-5*\left(3*\begin{bmatrix}-6 & 2\\7 & 1\end{bmatrix}-2*\begin{bmatrix}5 & 2\\8 & 1\end{bmatrix}+4*\begin{bmatrix}5 & -6\\8 & 7\end{bmatrix}\right)$$

Second row

$$-1*\begin{bmatrix}3 & 2 & 1\\5 & -6 & 0\\8 & 7 & 3\end{bmatrix} \qquad -1*\begin{bmatrix}3 & 2 & 1\\5 & -6 & 0\\8 & 7 & 3\end{bmatrix}$$

$$-1*\left(-5*\begin{bmatrix}2 & 1\\7 & 3\end{bmatrix}-6*\begin{bmatrix}3 & 1\\8 & 3\end{bmatrix}\right)$$

Once 2x2 minors are reached, no further recursion is possible.

Now do the following operation on each 2x2 minor:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Longrightarrow \text{A*D - B*C}$$

This results in the final calculation …

$$-5*[3(-6-14)-2(5-16)+4(35+48)]-1*[-5(6-7)-6(9-8)]$$
$$=-1469$$

General info on determinants

The following excerpt is from a study book written by Chris Harman and Patricia Cretchley, associate professors at the Faculty of Sciences in The University of Southern Queensland.

We know how to clear factor 3 in the equation 3x = 5. We simply multiply both sides by its multiplicative inverse or reciprocal, 1/3. We also say we divide by 3*.*
We can clear any factor except 0 in that way. 0 is the only real number that does not have a multiplicative inverse.
Let us apply a similar argument to matrix algebra: suppose we have matrix equation AX = B. Some square matrices A have multiplicative inverses, A^{-1}*, but many do not. This means that we cannot always get rid of a matrix factor A in a matrix equation. And that is why we do not define matrix division:* it simply can't always be done!
To be able to distinguish between those matrices that have inverses and those that do not, we now define a number called the determinant *of a matrix, so that*
- *when the determinant of the matrix is 0, the matrix has no inverse;*
- *when the determinant is not 0, the matrix is invertible (non-singular).*

The term 'determinant' was first introduced by Gauss in *Disquisitiones arithmeticae* (1801) while discussing quadratic forms. He used the term because the determinant determines the properties of the quadratic form.

Important properties of the determinant include the following, which include invariance under elementary row and column operations.

1. Switching two rows or columns changes the sign.
2. Scalars can be factored out from rows and columns.
3. Multiples of rows and columns can be added together without changing the determinant's value.
4. Scalar multiplication of a row by a constant *c* multiplies the determinant by *c*.
5. A determinant with a row or column of zeros has value 0.
6. Any determinant with two rows or columns equal has value 0.

Solution of linear simultaneous equations provided without proof

An example of solving 3 linear equations can be expressed as

$$a_1x + b_1y + c_1y + d_1 = 0$$
$$a_2x + b_2y + c_2y + d_2 = 0$$
$$a_3x + b_3y + c_3y + d_3 = 0$$

Using determinants this is solved by the following relationship

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Example

Solve the equation..

5x - 6y + 4z = 15
7x + 4y - 3z = 19
2x + y + 6z = 46

This equation is rewritten as

5x - 6y + 4z - 15 = 0
7x + 4y - 3z - 19 = 0
2x + y + 6z - 46 = 0

Expressing this in determinant form

$$\frac{x}{\begin{vmatrix} -15 & -6 & 4 \\ -19 & 4 & -3 \\ -46 & 1 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 5 & -15 & 4 \\ 7 & -19 & -3 \\ 2 & -46 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 & -15 \\ 7 & 4 & -19 \\ 2 & 1 & -46 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}}$$

On evaluating the denominators.

(x /-1257) = (y /-1676) = (z / -2514) = (-1 /419)

Dividing each denominator by 419 results in

(x /--3) = (y /-4) = (z / -6) = -1

This results in

x = 3: y =4: z =6

Cross Product

From earlier writing

The cross product of 2 vectors is the multiplication of 2 vectors whose origins are at the center. The result is a third vector perpendicular to the other 2. This perpendicular vector **uniquely identifies the plane (in 3D)** where the original vectors reside in.[1]

Given the vectors $\mathbf{OP_1} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{OP_2} = \langle x_2, y_2, z_2 \rangle$

The cross product of these two vectors result in a third vector,

$$OP_1 \times OP_2 = \langle y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2 \rangle$$

Note: The length of the vector obtained by the cross product of OP_1 and OP_2 is

$$|OP_1 \times OP_2| = |OP_1| * |OP_2| * \sin\theta$$

Where θ is the angle between OP_1 and OP_2 in the range $0 < \theta < \pi$

The following excerpt is from a study book written by Chris Harman and Patricia Cretchley, associate professors at the Faculty of Sciences in The University of Southern Queensland.

- *$\mathbf{v}\times\mathbf{u}$ has magnitude $|\mathbf{u}|\ |\mathbf{v}|\ \sin\varphi$ (like dot product but with sine instead of cos);*
- *$\mathbf{v} \times \mathbf{u}$ has the direction of the thumb in the right-hand rule;*
- *$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ surprisingly;*
- *the magnitude $|\mathbf{u} \times \mathbf{v}|$ also gives the area of a parallelogram;*
- *the absolute value of a scalar triple product $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$ gives the volume of a parallelepiped.*

following on from determinants

An interesting application using the procedure to calculate determinants is in the determining of the cross product of 2 vectors.

Consider vectors $\mathbf{a} = \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$

[1] The cross-product is used in animation to determine the angle of all the surfaces so that lighting can be applied proportionally. This technique of applying the cross product is called 'normalisation'.

Then the cross product of the 2 vectors is

$$\mathbf{a\,x\,b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{bmatrix} = \mathbf{i}\begin{bmatrix} a2 & a3 \\ b2 & b3 \end{bmatrix} - \mathbf{j}\begin{bmatrix} a1 & a3 \\ b1 & b3 \end{bmatrix} + \mathbf{k}\begin{bmatrix} a1 & a2 \\ b1 & b2 \end{bmatrix}$$

$$\mathbf{a\,x\,b} = \begin{bmatrix} a2b3 - a3b2 \\ a3b1 - a1b3 \\ a1b2 - a2b1 \end{bmatrix}\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

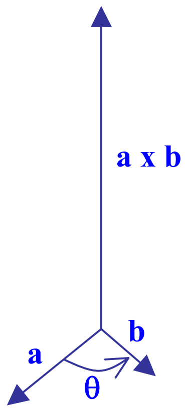
… where **i**, **j**, and **k** are unit vectors (having a length of 1) in the x, y, and z direction respectively.

More often then not these **i**, **j**, and **k** vectors are implied and thereby not written, resulting in

$$\mathbf{a\,x\,b} = \begin{bmatrix} a2b3 - a3b2 \\ a3b1 - a1b3 \\ a1b2 - a2b1 \end{bmatrix}$$

Knowing the procedure of finding the determinant is therefore also a great tool in calculating the cross product.

The vector obtained by applying the cross product on two known vectors is perpendicular to both those vectors …



If θ is the angle between two vectors **a** and **b**, with $0 \le \theta \le \pi$ then

$$|\mathbf{a\,x\,b}| = |\mathbf{a}||\mathbf{b}|\mathbf{sin\theta}$$

… where $|\mathbf{a\,x\,b}|$ is the area of the parallelogram determined by **a** and **b**.

Looking at the figure, the direction of the angle can be visualized by the **right-hand rule**:

*If the fingers of your right hand curl in the direction of the rotation (through an angle less than 180 degrees), from **a** to **b**, then your thumb points in the direction of **a** x **b**.*

Dot Product

From earlier writings

The dot product of 2 vectors multiplies the corresponding coordinates and adds the values.

For vectors $OA_1 = \langle x_1, y_1, z_1 \rangle$ and $OB = \langle x_2, y_2, z_2 \rangle$,

$$OA \cdot OB = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Note that the result of the dot product is a real number (a scalar) and that its operation symbol is a 'dot'. Contrast this with the cross product that results in another vector and whose operation symbol is a 'cross'.

The dot product is used to determine the angle between 2 vectors.

$$OA \cdot OB = |OA||OB| \cos\theta$$

$$\cos\theta = \frac{OA \cdot OB}{|OA||OB|}$$

In fact, **we can think of the dot product as measuring the extent to which the 2 vectors are pointing in the same direction**.

If OA and OB point in the same general direction,
$OA \cdot OB > 0$

If OA and OB point in the same general opposite direction,
$OA \cdot OB < 0$

If OA and OB point to exactly the opposite direction,
$\theta = \pi$ → $\cos \pi = -1$ → $OA \cdot OB = -|OA||OB|$

If OA and OB are perpendicular (orthogonal),
$\theta = \pi/2$ → $\cos \pi/2 = 0$ → $OA \cdot OB = 0$

It is this last property of the dot product that will prove to be useful to us.

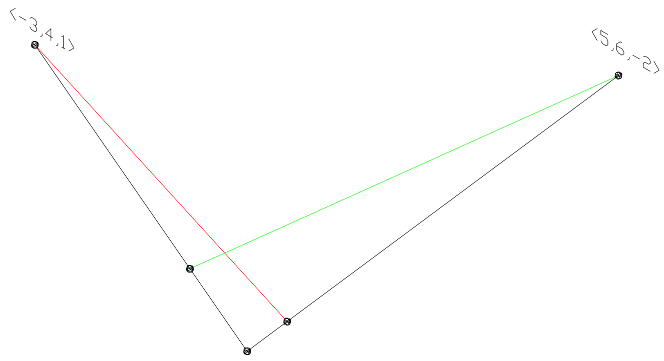
As an interesting aside, when programming, it is better to work with the square of the length whenever possible to avoid expensive root calculations.

Following on from determinants and cross product

When corresponding coordinates of 2 vectors are multiplied and then added together, we derive a scalar value that can be interpreted as the multiplication of the length of one vector with the length derived from the projection of the other vector onto it.
Following illustration will clarify this …

Draw vectors **a** and **b** in CAD

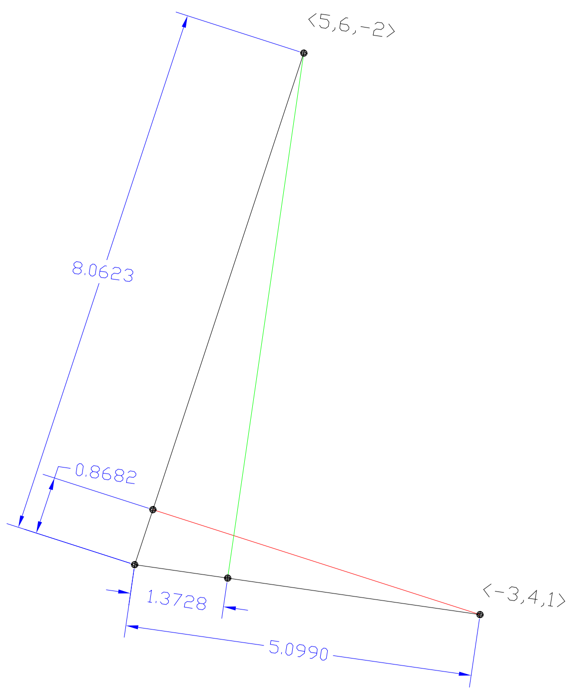
$$\mathbf{a} = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$



The illustration shows these vectors in SE isometric view.

Then, using CAD, draw the following perpendiculars:

- The red line being the projection of **b** onto **a**.
- The green line being the projection of **a** onto **b**.



To get true lengths in CAD, the coordinate axes are aligned with the face formed by the vectors **a** and **b**.

Calculate dot product using coordinates:

$$\mathbf{a.b} = \mathbf{b.a} = (5*-3)+(6*4)+(-2*1) = \mathbf{7}$$

Calculate dot product using multiplication of lengths of vectors with scalar projections:

8.0623 * 0.8682 = 7

5.0990 * 1.3728 = 7

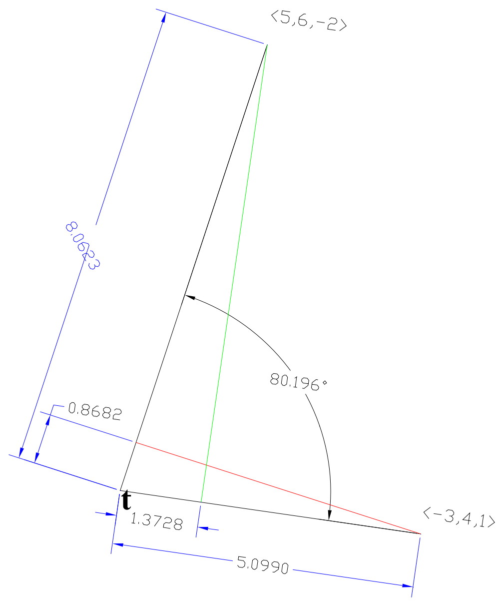
Note that this arrangement occurs a lot in physical situations; the perpendicular projection of one vector, A, onto another one, B, gives you the influence A has on B. An example would be the influence of A, being the direction and force of the wind, on B, the direction of the boat.

The dot product enables you to find angles between vectors in 3D:

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

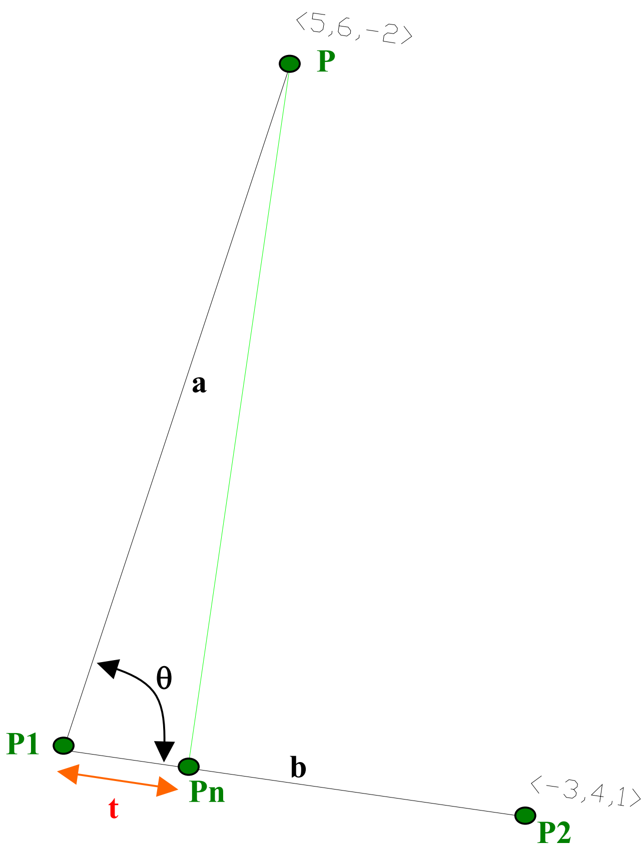
$$\mathbf{a.b} = 8.0623 * 5.0990 * \cos(80.196)$$

$$\mathbf{a.b} = 7$$



Application example: Find perpendicular to a line in 3D from a given point

This is an interesting application that illustrates the use of the dot product. Continuing our example from above, consider the point (5,6,-2) and the line from (0,0,0) to (-3,4,1).



We know that …

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

But note also from basic trigonometry that

$$|\mathbf{t}| = |\mathbf{a}|\cos\,\theta$$

Combining these 2 equations gives us …

$$|\mathbf{t}| = \frac{\mathbf{a.b}}{|\mathbf{b}|} \qquad |\mathbf{t}| = \frac{7}{5.1} = 1.3725$$

This agrees with the dimensions shown in the figure above.

Note that the dot product is the proportion factor between the 2 lengths |t| and |b|

If we set |**w**| to

$$|\mathbf{w}| = \frac{|\mathbf{t}|}{|\mathbf{b}|} \qquad |\mathbf{w}| = \frac{\mathbf{1.3725}}{\mathbf{5.1}} = \mathbf{0.26923}$$

Then the coordinates of the projection point Pn can be calculated with

$$\mathbf{Pn}_{[i]} = \mathbf{P1}_{[i]} + |\mathbf{w}|\left(\mathbf{P2}_{[i]} - \mathbf{P1}_{[i]}\right)$$

$$\mathbf{Pn}_{[x]} = \mathbf{0} + \mathbf{0.26923}(-\mathbf{3} - \mathbf{0})$$

$$\mathbf{Pn}_{[y]} = \mathbf{0} + \mathbf{0.26923}(4 - \mathbf{0})$$

$$\mathbf{Pn}_{[z]} = \mathbf{0} + \mathbf{0.26923}(\mathbf{1} - \mathbf{0})$$

$$\mathbf{Pn} = \begin{bmatrix} -\mathbf{0.80769} \\ \mathbf{1.07692} \\ \mathbf{0.26923} \end{bmatrix}$$

Example with different center

consider points
P = (5, 16, -8)

and line through points

P1 = (-2, -3, 4)

P2 = (11, 6, -1)

length of the line P1_P2 =

$$|\mathbf{P1\_P2}| = \sqrt{(\mathbf{x2} - \mathbf{x1})^2 + (\mathbf{y2} - \mathbf{y1})^2 + (\mathbf{z2} - \mathbf{z1})^2}$$

$$|\mathbf{P1\_P2}| = \sqrt{(\mathbf{11} + \mathbf{2})^2 + (\mathbf{6} + \mathbf{3})^2 + (-\mathbf{1} - \mathbf{4})^2} = \sqrt{\mathbf{275}} = \mathbf{16.583}$$

$$|\,\mathbf{t}\,| = \frac{\mathbf{P.P2}}{|\mathbf{P1\_P2}|} = \frac{(\mathbf{x} - \mathbf{x1})(\mathbf{x2} - \mathbf{x1}) + (\mathbf{y} - \mathbf{y1})(\mathbf{y2} - \mathbf{y1}) + (\mathbf{z} - \mathbf{z1})(\mathbf{z2} - \mathbf{z1})}{\mathbf{16.583}}$$

$$|\,\mathbf{t}\,| = \frac{(\mathbf{5} + \mathbf{2})(\mathbf{11} + \mathbf{2}) + (\mathbf{16} + \mathbf{3})(\mathbf{6} + \mathbf{3}) + (-\mathbf{8} - \mathbf{4})(-\mathbf{1} - \mathbf{4})}{\mathbf{16.583}} = \mathbf{19.417}$$

$$|\,\mathbf{w}\,| = \frac{|\mathbf{t}|}{|\mathbf{P1\_p2}|}$$

$$|\,\mathbf{w}\,| = \frac{\mathbf{19.417}}{\mathbf{16.583}} = 1.171$$

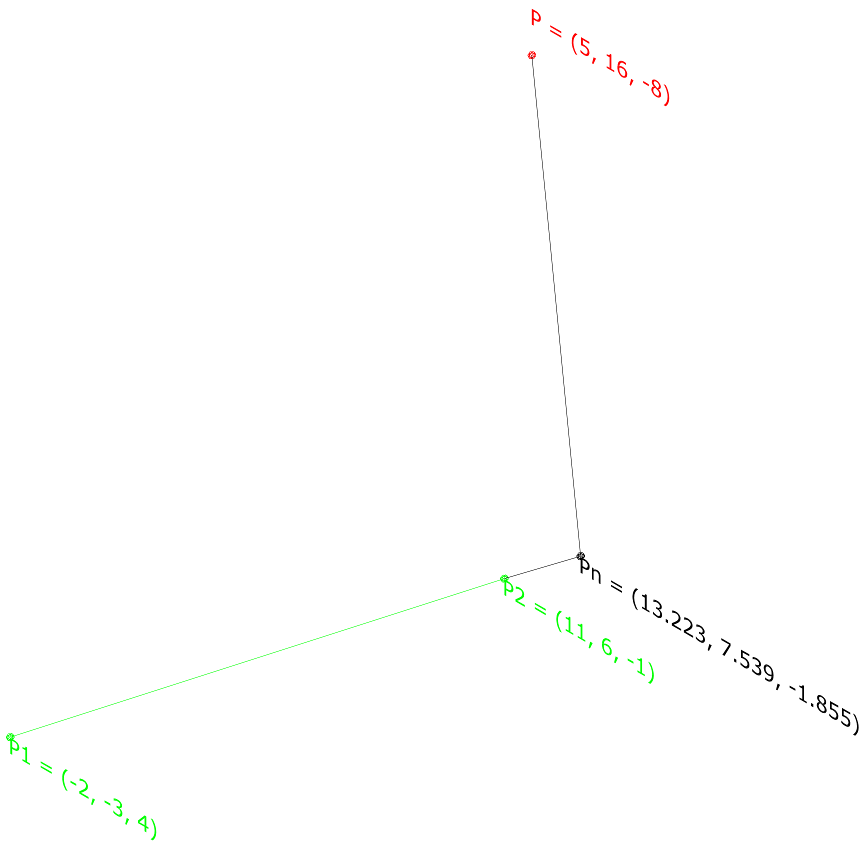
$$\mathbf{Pn}_{[\mathbf{i}]} = \mathbf{P1}_{[\mathbf{i}]} + |\mathbf{w}|\left(\mathbf{P2}_{[\mathbf{i}]} - \mathbf{P1}_{[\mathbf{i}]}\right)$$

$$\mathbf{Pn}_{[x]} = -2 + 1.171(11 - -2) = 13.223$$

$$\mathbf{Pn}_{[y]} = -3 + 1.171(6 - -3) = 7.539$$

$$\mathbf{Pn}_{[z]} = 4 + 1.171(-1 - 4) = -1.855$$

$$Pn = \begin{bmatrix} 13.223 \\ 7.539 \\ -1.855 \end{bmatrix}$$



The Barycenter

Barycentric combinations are a way to extract out of n points a resulting point Q in a way independent of the origin.

To calculate Q
- Take the sum of all points in which each one is scaled with a real factor (weight factor).
- The weight factor of each point determines how much that point affects the sum.
- Therefore the weight factor tells you how much Q is influenced by that point.

General equation for Barycentric combinations

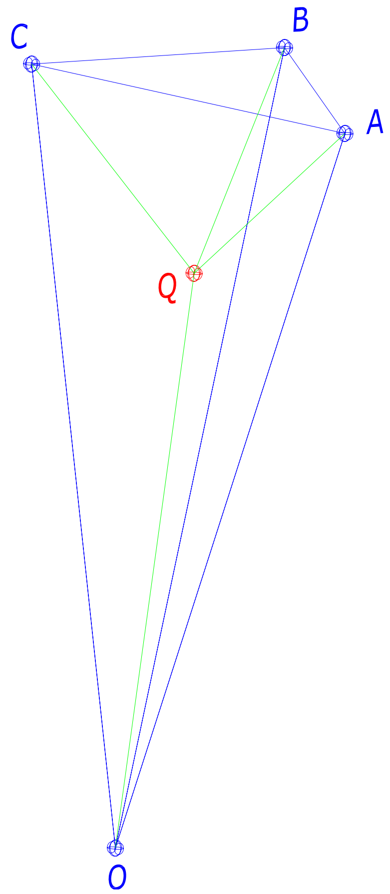
$$Q = \sum_{i=0}^{n-1} (W_{[i]} * P_{[i]}) \ with \ \sum_{i=0}^{n-1} (W_{[i]}) = 1$$

Center of mass (special case of the above)

$$Q = \frac{1}{n} \sum_{i=0}^{n-1} P_{[i]}$$

Example 1

Given a tetrahedron with following coordinates …



$$O = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} \qquad A = \begin{bmatrix} 0.0 \\ 9.3417 \\ 3.5682 \end{bmatrix}$$

$$B = \begin{bmatrix} 2.0177 \\ 9.7943 \\ 0.0 \end{bmatrix} \qquad C = \begin{bmatrix} -2.0177 \\ 9.7943 \\ 0.0 \end{bmatrix}$$

Calculate Q …

$$Q_{[x]} = \frac{1}{4}(0.0 + 0.0 + 2.0177 - 2.0177) = 0.0$$

$$Q_{[y]} = \frac{1}{4}(0.0 + 9.3417 + 9.7943 + 9.7943) = 7.232575$$

$$Q_{[z]} = \frac{1}{4}(0.0 + 3.5682 + 0.0 + 0.0) = 0.89205$$

$$Q = \begin{bmatrix} 0.0 \\ 7.232575 \\ 0.89205 \end{bmatrix}$$

Example 2

Given a truncated tetrahedron with coordinates …

$$A = \begin{bmatrix} 0.0 \\ 9.3417 \\ 3.5682 \end{bmatrix} \quad B = \begin{bmatrix} 2.0177 \\ 9.7943 \\ 0.0 \end{bmatrix} \quad C = \begin{bmatrix} -2.0177 \\ 9.7943 \\ 0.0 \end{bmatrix}$$

$$At = \begin{bmatrix} 0.0 \\ 7.0063 \\ 2.6762 \end{bmatrix} \quad Bt = \begin{bmatrix} 1.5133 \\ 7.3457 \\ 0.0 \end{bmatrix} \quad Ct = \begin{bmatrix} -1.5133 \\ 7.3457 \\ 0.0 \end{bmatrix}$$

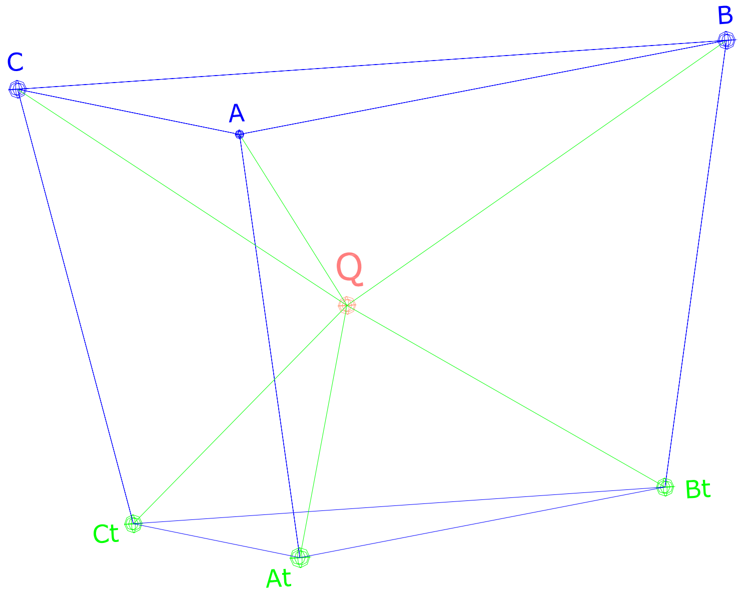
Calculate Q …

$$Q_{[x]} = \frac{1}{6}(0.0 + 2.0177 - 2.0177 + 0.0 + 1.5133 - 1.5133) = 0.0$$

$$Q_{[y]} = \frac{1}{6}(9.3417 + 9.7943 + 9.7943 + 7.0063 + 7.3457 + 7.3457) = 8.438$$

$$Q_{[z]} = \frac{1}{6}(3.5682 + 0.0 + 0.0 + 2.6762 + 0.0 + 0.0) = 1.0407$$

$$Q = \begin{bmatrix} 0.0 \\ 8.438 \\ 1.0407 \end{bmatrix}$$



Example 3

Given the truncated tetrahedron given above, we can scale this object by applying a ratio to each vertex - center length, thereby deriving coordinates of a new tetrahedron which is scaled around the center.

Assuming a scaling of 1/5th, Barycentric center Q, and vertex U, the coordinates are …

$$Us_{[i]} = \frac{4U_{[i]} + Q_{[i]}}{5}$$

$$As_{[i]} = \frac{4A_{[i]} + Q_{[i]}}{5} = \begin{bmatrix} 0.0 \\ 9.16096 \\ 3.0627 \end{bmatrix} \qquad Ap_{[i]} = \frac{4At_{[i]} + Q_{[i]}}{5} = \begin{bmatrix} 0.0 \\ 7.292632 \\ 2.349068 \end{bmatrix}$$

$$Bs_{[i]} = \frac{4B_{[i]} + Q_{[i]}}{5} = \begin{bmatrix} 1.61416 \\ 9.52304 \\ 0.20814 \end{bmatrix} \qquad Bp_{[i]} = \frac{4Bt_{[i]} + Q_{[i]}}{5} = \begin{bmatrix} 1.21064 \\ 7.56416 \\ 0.20814 \end{bmatrix}$$

$$Cs_{[i]} = \frac{4C_{[i]} + Q_{[i]}}{5} = \begin{bmatrix} -1.61416 \\ 9.52304 \\ 0.20814 \end{bmatrix} \qquad Cp_{[i]} = \frac{4Ct_{[i]} + Q_{[i]}}{5} = \begin{bmatrix} -1.21064 \\ 7.564192 \\ 0.20814 \end{bmatrix}$$